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The Chern–Simons term induced at high temperature and the quantization of its coefficient

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Abstract. By perturbative calculations of the high-temperature ground-state axial vector current of fermion fields coupled to gauge fields, an anomalous Chern–Simons topological mass term is induced in the three-dimensional effective action. The anomaly in three dimensions appears just in the ground-state current rather than in the divergence of ground-state current. In the Abelian case, the contribution comes only from the vacuum polarization graph, whereas in the non-Abelian case, contributions come from the vacuum polarization graph and the two triangle graphs. The relation between the quantization of the Chern–Simons coefficient and the Dirac quantization condition of magnetic charge is also obtained. It implies that in a (2+1)-dimensional QED with the Chern–Simons topological mass term and a magnetic monopole with magnetic charge g present, the Chern–Simons coefficient must be also quantized, just as in the non-Abelian case.

1 Introduction

It is well known that there are anomalous terms in the Ward identities for axial currents in four-dimensional space-time [1]. By perturbative calculations for the triangle diagrams of axial vector currents in fermion fields coupled to gauge fields, it is found that there is an additional anomalous term in the divergence of axial vector current [2], if we make the divergence of the vector current keep conservation. This leads us to the question: What happens in three-dimensional space theories? Here we shall answer this question.

In this paper, within the framework of the imaginary time formalism of finite-temperature gauge field theory [3], we make the perturbative calculations for a ground-state axial vector current in four-dimensional space-time at high-temperature limit, i.e., the time compactification; the four-dimensional space is then reduced to the three-dimensional space, and we therefore get an additional anomalous term in the three-dimensional ground-state axial vector current. Moreover, we get an anomalous Chern–Simons topological mass term [4–6] in the three-dimensional effective action.

We consider the radiative generation of a Chern–Simons topological mass term in the high-temperature effective action of a gauge field theory coupled to fermion fields. It is known that the fermionic current axial vector is partially conserved and that there are anomalies of the current axial vector.

The Chern–Simons topological mass term is gauge-invariant under large (nontrivial) gauge transformations only if its coefficient is quantized [5]. In the non-Abelian case, for any semisimple Lie group G, the gauge transformations are characterized by the topological classes labeled by $\Pi_3(G) = Z$, corresponding to the mapping of S_3 of compactified three-dimensional space to the group G [5,7]. Under the gauge transformations with a nonvanishing winding number, the Chern–Simons action is not invariant, but for invariance of the exponentiated Chern–Simons action to be ensured, the Chern–Simons coefficient must be quantized. In the Abelian case, there are no instantons [8], but in the existence of a magnetic monopole, the Chern–Simons coefficient must be also quantized.

The paper is organized as follows. In the next section, the Chern–Simons term is induced at high temperature. In Sect. 3, we describe and discuss the quantization of Chern–Simons coefficient. Conclusions are presented in Sect. 4.

2 The Chern–Simons term induced at high temperature

Let us start with the ground-state current axial vector of Dirac fields in four-dimensional space-time [9]:

$$\begin{aligned}\langle 0|J_{\mu 5}(x)|0\rangle &= \langle 0|\frac{1}{2}[\bar{\Psi}(x), \gamma_\mu\gamma_5\Psi(x)]_-|0\rangle \\ &= -\mathrm{Tr}[\gamma_\mu\gamma_5 G(x,x')]|_{x'\to x} \qquad (1)\end{aligned}$$

where $G(x,x')$ is the Dirac propagator in interaction with an electromagnetic field $A_\mu(x)$ and satisfies following equation:

$$(\not{D} - m)G(x,x') = \delta^4(x-x') \qquad (2)$$

with

$$\not{D} = \mathrm{i}\not{\partial} - e\not{A} = \gamma_\mu(\mathrm{i}\partial_\mu - eA_\mu(x)). \qquad (3)$$

Within the imaginary time formalism of finite-temperature gauge-field theory, the ground-state current axial vector can be expanded perturbatively at one-loop level in momentum space:

$$\begin{aligned}
&\int_- \frac{\mathrm{d}^4p_1}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu\gamma_5 G(p_1)](2\pi)^3\beta\delta^3(\vec{p}-\vec{p}')\delta_{\omega_n-\omega_{n'}} \\
&= \int_- \frac{\mathrm{d}^4p_1}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu\gamma_5 G_0(p_1)](2\pi)^3\beta\delta^3(\vec{p}-\vec{p}')\delta_{\omega_n-\omega_{n'}} \\
&\quad -e\int_- \frac{\mathrm{d}^4p_1}{(2\pi)^4}\int_+ \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu\gamma_5 G_0(p_1)\not{A}(k)G_0(p_1-k)] \\
&\quad \times(2\pi)^3\beta\delta^3(\vec{p}-\vec{p}'-\vec{k})\delta_{\omega_n-\omega_{n'}-\omega_m} \\
&\quad +e^2\int_- \frac{\mathrm{d}^4p_1}{(2\pi)^4}\int_+ \frac{\mathrm{d}^4k_1}{(2\pi)^4}\int_+ \frac{\mathrm{d}^4k_2}{(2\pi)^4} \\
&\quad \times\mathrm{Tr}[\gamma_\mu\gamma_5 G_0(p_1+k_2)\not{A}(k_2)G_0(p_1)\not{A}(k_1)G_0(p_1-k_1)] \\
&\quad \times(2\pi)^3\beta\delta^3(\vec{p}-\vec{p}'-\vec{k}_1-\vec{k}_2)\delta_{\omega_n-\omega_{n'}-\omega_{m_1}-\omega_{m_2}} \\
&\quad -e^3\int_- \frac{\mathrm{d}^4p_1}{(2\pi)^4}\int_+ \frac{\mathrm{d}^4k_1}{(2\pi)^4}\int_+ \frac{\mathrm{d}^4k_2}{(2\pi)^4}\int_+ \frac{\mathrm{d}^4k_3}{(2\pi)^4} \\
&\quad \times\mathrm{Tr}\left[\gamma_\mu\gamma_5 G_0(p_1+k_3)\not{A}(k_3)G_0(p_1)\not{A}(k_2)\right. \\
&\quad \left.\times G_0(p_1-k_2)\not{A}(k_1)G_0(p_1-k_2-k_1)\right] \\
&\quad \times(2\pi)^3\beta\delta^3(\vec{p}-\vec{p}'-\vec{k}_1-\vec{k}_2-\vec{k}_3) \\
&\quad \times\delta_{\omega_n-\omega_{n'}-\omega_{m_1}-\omega_{m_2}-\omega_{m_3}} \\
&\quad +\cdots \qquad (4)
\end{aligned}$$

with

$$\int_- \frac{\mathrm{d}^4p}{(2\pi)^4} = \frac{1}{\beta}\sum_n\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}, p_0=\omega_n = \frac{(2n+1)\pi}{\beta}, \quad \text{for fermions,} \qquad (5)$$

and

$$\int_+ \frac{\mathrm{d}^4k}{(2\pi)^4} = \frac{1}{\beta}\sum_m\int\frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}, k_0=\omega_m = \frac{2m\pi}{\beta}, \quad \text{for bosons,} \qquad (6)$$

$$(2\pi)^4\delta^4(p-p') = (2\pi)^3\delta^3(\vec{p}-\vec{p}')\beta\delta_{\omega_n-\omega_{n'}} \qquad (7)$$

where $G_0(p)$ is a free Dirac propagator, $T=1/\beta$ is the temperature, and $m,n=0,\pm1,\pm2,\pm3,....$

At the high-temperature limit, $T\to\infty$, i.e., the time compactification, the four-dimensional space-time is reduced to the three-dimensional space; therefore, the superficial degrees of divergence in the integrands of the right side of (4) are also decreased, and then only the vacuum polarization graph corresponding to the second term of the right side of (4) and the triangle graphs corresponding to the third term of the right side of (4) are divergent. Thus the axial anomalies in (4) are from the vacuum polarization graph and the triangle graphs. Notice that the contribution of the first term of right side of (4) vanishes; this is because these axial anomalies come from only the one-loop diagrams, there is no axial anomaly at two-loop and beyond two-loop levels, and no gauge propagator appears in the fermionic one-loop diagrams. So the gauge field $A_\mu(x)$ can be regarded as a background classical field.

Now we consider the vacuum polarization graph and the triangle graphs. At high temperature, since the temporal component of the gauge potential $A_0(x)$ decouples [10,11], the temporal component of the ground-state current axial vector, $\langle 0|J_{05}(x)|0\rangle$, does not contribute to the effective action, $I_{\mathrm{eff}}(A)$, because of the relation

$$\frac{\delta I_{\mathrm{eff}}(A)}{\delta eA_\mu(x)} = \langle 0|J_{\mu5}(x)|0\rangle. \qquad (8)$$

At high temperature, fermions behave as massless; thus we can neglect the fermion mass m. Then the algebra of these γ matrices may be realized by the three two-dimensional Pauli matrices. Therefore

$$\gamma_j = \mathrm{i}\sigma_j, j=1,2,3, \gamma_0 = I \text{ (identity)}, \gamma_5=\sigma_1\sigma_2\sigma_3. \qquad (9)$$

First we calculate the contribution coming from the vacuum polarization graph. We denote

$$\Pi^{\mu\nu}(k) = \int_- \frac{\mathrm{d}^4p_1}{(2\pi)^4}\mathrm{Tr}[\gamma_\mu\gamma_5 G_0(p_1-k)]. \qquad (10)$$

Since what we require is the effective action, the γ_μ and γ_ν in $\Pi^{\mu\nu}(k)$ are to couple to gauge potentials $A_\mu(x)$ and $A_\nu(x)$ if we are considering (8). At high temperature, there are only the spatial components in $\Pi^{\mu\nu}(k)$ of (10) because of the decoupling of $A_0(x)$, and only $n,m=0$ zero modes in (5) and (6) are taken. Therefore we have obtained

$$\Pi^{ij}(\vec{k}) = \frac{T}{4\pi}\epsilon^{ijl}k_l, \qquad (11)$$

where the indices i,j,l are taken over $1,2,3$, and ϵ^{ijl} is a three-dimensional totally antisymmetric tensor. Although there are ϵ^{ijl} and γ_5 when calculations for $\Pi^{\mu\nu}(k)$ are done in (10), the dimensional regularization can be used for $\Pi^{\mu\nu}(k)$ because of the one-loop level [12,13]; the intrinsic regularization method proposed in the literature [14] can also be used, without the difficulties brought by the definitions of γ_5 and ϵ_{ijl} in three dimensions, but the results are the same. Here we have renormalized for zero temperature, since the renormalization of ultraviolet divergence at finite temperature is the same with one at zero temperature [15].

We define $e'=\sqrt{T}e$, where e' is the three-dimensional coupling constant with dimension $[mass]^{1/2}$ in the Abelian case, and e is the four-dimensional coupling constant that is dimensionless in Abelian case. We also define the three-dimensional gauge potentials $A_j(\vec{k})=\sqrt{T}A_j(\vec{k},k_0=0)$ in momentum space in Abelian case, but $A_j(\vec{k},k_0=0)$ are four-dimensional, where $A_j(\vec{k})$ have dimension $[mass]^{-5/2}$ and $A_j(\vec{k},k_0=0)$ have dimension $[mass]^{-3}$. In the Abelian case, the two triangle graphs lead to contributions of opposite sign to the high temperature ground-state current; therefore their contributions cancel each other. Thus, only the vacuum polarization diagram contributes to the high-temperature ground-state current in the three-dimensional space in the Abelian case, and the result is

$$\frac{e'}{8\pi}\epsilon^{ijl}[\partial_j A_l(\vec{x}) - \partial_l A_j(\vec{x})], \quad (12)$$

where $A_j(\vec{x})$ are the Fourier transformations of $A_j(\vec{k})$. In the non-Abelian case, the contributions to the high-temperature ground state current come from the vacuum polarization graph and the two triangle graphs. They are

$$\frac{\epsilon^{ijl}}{8\pi}\mathrm{Tr}F_{jl}(\vec{x}) \quad (13)$$

with

$$F_{jl}(\vec{x}) = \partial_j A_l(\vec{x}) - \partial_l A_j(\vec{x}) + [A_j(\vec{x}), A_l(\vec{x})]_- \quad (14)$$

$$A_j(\vec{x}) = g' T^b A_j^b(\vec{x}), \quad (15)$$

where T^b are generators of the gauge group, and $g' = \sqrt{T}g$ is the three-dimensional coupling constant of dimension $[mass]^{1/2}$ in which g is a dimensionless coupling constant in four-dimensional space-time.

In view of the variational relation between the effective action and the ground state current, we have derived the induced three-dimensional effective action in the high-temperature limit for the non-Abelian case to be

$$\begin{aligned} I_{\mathrm{eff}}^{\mathrm{CS}}(A) &= \frac{\epsilon^{ijl}}{8\pi}\int \mathrm{d}^3\vec{x}\,\mathrm{Tr}\left[\frac{1}{2}F_{ij}(\vec{x})A_l(\vec{x}) \right.\\ &\quad \left. - \frac{1}{3}A_i(\vec{x})A_j(\vec{x})A_l(\vec{x})\right] \\ &= \pi W(A). \end{aligned} \quad (16)$$

This is an anomalous three-dimensional effective action; however, the conventional action in the three-dimensional gauge theory is the original Yang–Mills action, i.e.,

$$I_{\mathrm{eff}}^{\mathrm{YM}}(A) = -\int \mathrm{d}^3\vec{x}\frac{1}{2(g')^2}\mathrm{Tr}[F^{jl}(\vec{x})F_{jl}(\vec{x})]. \quad (17)$$

As one knows, the dimension of gauge potentials, $A_\mu^a(x)$, is

$$[A_\mu^a(x)] = \frac{d-2}{2}. \quad (18)$$

For the three-dimensional theory, $d = 3$, thus the dimension of $A_j^a(\vec{x})$ is $[A_j^a(\vec{x})] = 1/2$. After multiplying $W(A)$ in (16) by $(2\pi\mu)/((g')^2)$, and adding it to (17), we obtain the effective action to be

$$\begin{aligned} &I_{\mathrm{eff}}(A) \\ &= \int \mathrm{d}^3\vec{x}\{\frac{1}{2(g')^2}\mathrm{Tr}[F^{jl}(\vec{x})F_{jl}(\vec{x})] \\ &\quad -\frac{\mu}{8\pi(g')^2}\epsilon^{ijl}\mathrm{Tr}[F_{ij}(\vec{x})A_l(\vec{x}) - \frac{2}{3}A_i(\vec{x})A_j(\vec{x})A_l(\vec{x})]\}. \end{aligned} \quad (19)$$

It can be seen from the dimension analysis that the parameter μ in (19) has the dimension of mass. The second term in (19) is the three-dimensional Chern–Simons topological mass term, which violates parity conservation because of the tensor ϵ_{ijl} and produces a mass for gauge fields. The $W(A)$ in (16) is the Chern–Simons secondary characteristic class [16].

3 Quantization of the Chern–Simons coefficient

The effective action in (19) is not invariant under the nontrivial (large) gauge transformations, but if the Chern–Simons coefficient in (19) is quantized [5], the effective action is invariant under a large gauge transformation. Recently, the quantization of the Chern–Simons coefficient has attracted considerable attention [17–19]. The action with an added Chern–Simons topological mass term to the usual Yang–Mills gauge theory in a (2+1)-dimensional space-time remains invariant under small gauge transformations, but for invariance of the exponentiated action under large gauge transformations to be ensured, the coefficient of Chern–Simons topological mass term has to be quantized [5].

In the Abelian case [20], however, the Chern–Simons coefficient is in general not quantized in the absence of a topological charge; but the Chern–Simons coefficient must be also quantized in the presence of a topological charge, e.g., a magnetic pole, just as in the non-Abelian case. In the following we will demonstrate this.

Using the two-loop perturbative radiative corrections for the fermionic current vector in QED with a Chern–Simons topological mass term in a (2+1)-dimensional space-time, we look for the relation between the Dirac quantization condition [21] of a magnetic charge and the quantization of the Chern–Simons coefficient; thus the Dirac quantization condition of a magnetic charge leads to the quantization of the Chern–Simons coefficient.

Let us consider the following Lagrangian in the (2+1)-dimensional QED with a Chern–Simons mass term,

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4}F_{\mu\nu}^2 - \frac{m}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 \\ &+\bar{\Psi}[\gamma_\mu(\mathrm{i}\partial^\mu - eA^\mu) - m_\mathrm{f}]\Psi, \end{aligned} \quad (20)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; m, λ and m_f are the photon topological mass, a parameter of the gauge-fixing term, and the fermion mass, respectively. The space-time is Minkowski with signature $(+,-,-)$. $\epsilon^{\mu\nu\lambda}$ is a three-dimensional antisymmetric tensor. Our purpose is to find the relation between the quantization of the Chern–Simons coefficient and the Dirac quantization condition of a magnetic charge, in the presence of a magnetic pole in the Abelian case; we therefore start with the electron current vector $J_\mu(x)$ in an external electromagnetic field $A_\mu(x)$ with the Chern–Simons topological mass term in 2+1 dimensions.

The ground-state current vector of fermion fields in a (2+1)-dimensional space-time is

$$\begin{aligned} \langle 0|J_\mu(x)|0\rangle &= \langle 0|\frac{1}{2}[\bar{\Psi}(x), \gamma_\mu\Psi(x)]_-|0\rangle \\ &= -\mathrm{Tr}[\gamma_\mu G(x,x')]|_{x'\to x}, \end{aligned} \quad (21)$$

where $G(x,x')$ is the fermion propagator in interaction with an external electromagnetic field $A_\mu(x)$ which has the Chern–Simons mass term and satisfies the following equation:

$$(\not{D} - m_\mathrm{f})G(x,x') = \delta^3(x-x') \quad (22)$$

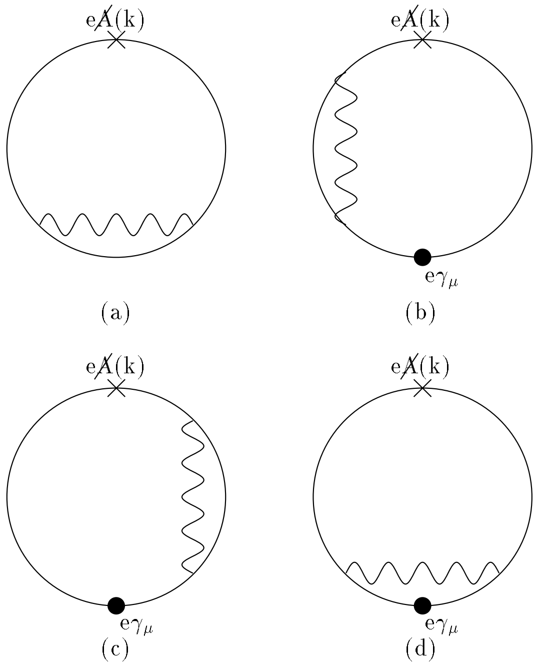

Fig. 1. The two-loop Feynman diagrams for $\langle 0|J_\mu|0\rangle$. The solid line stands for a fermion propagator, the wavy line stands for a gauge propagator in the pure Chern–Simons theory ($D_{\mu\nu}(k)$ in (24)) the cross for $e\not{A}$, and the black dot for $e\gamma_\mu$

with

$$\not{D} = \mathrm{i}\not{\partial} - e\not{A} = \gamma_\mu[\mathrm{i}\partial_\mu - eA_\mu(x)]. \tag{23}$$

What we are considering are the effects of the Chern–Simons mass term, so the photon propagator $D_{\mu\nu}$ is taken in the following form, which is produced by the pure Chern–Simons term [22], i.e., the second term of the Lagrangian in (20):

$$D_{\mu\nu} = \frac{\epsilon_{\mu\nu\lambda}k^\lambda}{mk^2}, \tag{24}$$

where it is in momentum space, and we adopt the Landau gauge $\lambda = 0$ in (20) in order to avoid the infrared divergences [23]. In the (2+1)-dimensional perturbative QED, the pure Chern–Simons effects brought by the fermion vector current in (21) are in the two-loop corrections in the lowest order, so we have to consider the two-loop Feynman diagrams for the ground-state current vector of the fermion fields, $\langle 0|J_\mu|0\rangle$, in a (2+1)-dimensional space-time in (21). The gauge propagators the appear in the two-loop diagrams, and therefore the gauge fields are regarded here as dynamical. These two-loop Feynman diagrams in momentum space are the diagrams (a), (b), (c), and (d) in Fig. 1.

The contribution of Fig. 1(a) to the ground-state current vector of the fermion fields, $\langle 0|J_\mu|0\rangle$, has the power of e^3, but the contributions of Fig. 1b,c,d to $\langle 0|J_\mu|0\rangle$ have the power of e^4. After calculating the contribution of Fig. 1(a) to the fermion current, we discover that it vanishes. Therefore, in the perturbative QED of 2+1 dimensions, the lowest order of the nonvanishing contributions with the effect of Chern–Simons term on the ground-state current vector of the fermion fields is $O(e^4)$, and the corresponding Feynman diagrams are the three diagrams (b), (c), and (d) in Fig. 1.

We now consider the contributions of the three diagrams (b), (c), and (d) in Fig. 1. We first look at the subdiagram in Fig. 1b, the electron self-energy. Its contribution in momentum space is

$$\begin{aligned}\Sigma(p) = {} & \frac{e^2}{4\pi m}(p^2 - m_f\not{p}) \\ & \times \int_0^1 \alpha^{\frac{1}{2}}[(1-\alpha)p^2 + m_f^2]^{-\frac{1}{2}}\mathrm{d}\alpha + 2m_f\end{aligned} \tag{25}$$

where the Landau gauge has been adopted, and the photon propagator has been taken as the form in (24), in accordance with the pure Chern–Simons theory, and the dimensional regularization has also been used. We compute the contribution $\Pi^{(\mathrm{b})}_{\mu\nu}(k)A_\nu(k)$ of Fig. 1(b) in terms of the electron self-energy $\Sigma(p)$ of (25). Because we are concerned with the pure Chern–Simons effects in $\Pi^{(\mathrm{b})}_{\mu\nu}(k)$, for simplicity, the electron mass m_f in $\Pi_{\mu\nu}(k)$ can be omitted, assuming the gauge boson mass $m \gg m_f$. After a tedious calculation, we get for $\Pi^{(\mathrm{b})}_{\mu\nu}(k)$:

$$\Pi^{(\mathrm{b})}_{\mu\nu}(k) = \frac{e^4}{48\pi^2 m}\left[-\frac{2}{\varepsilon} - \ln\frac{k^2}{\mu^2} - \gamma + O(\varepsilon)\right]\epsilon_{\mu\nu\lambda}k^\lambda, \tag{26}$$

where the dimensional regularization has been used and $\varepsilon = 3 - d$, dimension $d \to 3$, γ is a Euler constant, and μ is an arbitrary mass scale.

The two diagrams (b) and (c) in Fig. 1 obviously give equal contributions. We now turn to the diagram (d) in Fig. 1. Similarly, we proceed to the cumbersome computation of the contribution $\Pi^{(\mathrm{d})}_{\mu\nu}(k)A_\nu(k)$ of Fig. 1(d). to the ground-state vector current of the electron fields. We have also omitted the electron mass m_f in $\Pi^{(\mathrm{d})}_{\mu\nu}(k)A_\nu(k)$. We compute the contribution $\Pi^{(\mathrm{d})}_{\mu\nu}(k)$ in terms of the Ward–Takahashi identities between the electron self-energy and the vertex function, which is in the subdiagram of Fig. 1(d). After a tedious computation for $\Pi^{(\mathrm{d})}_{\mu\nu}(k)$, we obtain

$$\Pi^{(\mathrm{d})}_{\mu\nu}(k) = \frac{e^4}{12\pi^2 m}\left[-\frac{2}{\varepsilon} - \ln\frac{k^2}{\mu^2} - \gamma + O(\varepsilon)\right]\epsilon_{\mu\nu\lambda}k^\lambda \tag{27}$$

where $3 - d = \varepsilon$, $(k^2)^{\varepsilon/2} = 1 + (\varepsilon/2)\ln k^2 + \cdots$, and $\Gamma(\frac{3-d}{2}) = -\frac{2}{\varepsilon} - \gamma + \cdots$. Ultimately, after adding the contributions of the three diagrams in Fig. 1 to the fermion vector current J_μ, we have obtained the total contributions of Fig. 1(b), (c), (d) to the electron vector current J_μ in momentum space, i.e., the two-loop radiative corrections; therefore,

$$J_\mu(k) = \frac{e^4}{8\pi^2 m}\left[\frac{2}{\varepsilon} + \ln\frac{k^2}{\mu^2} + \gamma + O(\varepsilon)\right]\epsilon_{\mu\lambda\nu}k^\lambda A^\nu(k). \tag{28}$$

For the Coleman–Hill result [24], in a massive matter-coupled Abelian Chern–Simons theory, no higher loops contribute to the Chern–Simons term; only one-loop correction does. However, when the matter is massless, the Chern–Simons term not only has the one-loop contribution but also has contributions from two loops and beyond [25]. In our paper, electrons are massless, taking $m_f = 0$, so there are two-loop corrections to the Chern–Simons term. This is consistent with other works [24,25].

We now transform from a momentum space into a position space to consider (28). The procedure of the Fourier transformation to position space has been regularized so that there is no infrared divergence because the power of the momenta p in the denominator is less than four. In the position space, the magnetic field is a pseudoscalar, not a vector, and has only one component. Its direction is perpendicular to the two-dimensional plane. In position space, the magnetic field is

$$B = \vec{\partial} \times \vec{A}(x) = \epsilon_{ij}\partial_i A_j(x) \tag{29}$$

where $\epsilon_{ij} = \epsilon_{0ij}, i, j = 1, 2$.

We now topologically map a plane onto a two-dimensional unity sphere S_2. The magnetic flux through the sphere is

$$\oint_{S_2} B\mathrm{d}\sigma = \int B\mathrm{d}^2 x = g \tag{30}$$

by definition of the magnetic charge g contained inside the sphere. There is a magnetic monopole with a magnetic charge g.

The contributions of the self-energy counterterms and the vertex counterterms corresponding to Fig. 1b,c,d can be also readily computed. Here we have adopted the minimal subtracted renormalization scheme. After having considered the contributions of these counterterms and renormalized for the quantities in (28), we finally get the result in configuration space,

$$\int J_0(x)\mathrm{d}\vec{x} = e_r = \frac{e_r^4}{8\pi^2 m_r}\int \vec{B}\cdot \mathrm{d}\vec{x} = \frac{e_r^4}{8\pi^2 m_r} g \tag{31}$$

where the zeroth dividual quantity $J_0(x)$ of $J_\mu(x)$ is the electric charge density, and e_r and m_r are a renormalized electron charge and a renormalized Chern–Simons topological mass, respectively. This yields

$$\frac{1}{2\pi}(eg)_r = n, \tag{32}$$

$$4\pi(\frac{m}{e^2})_r = n, \tag{33}$$

where $n = 0, 1, 2, 3, \cdots$. Equation (32) is called the Dirac quantization condition of magnetic charge. Equation (33) implies the quantization of the coefficient of the Chern–Simons term. In Redlich's work [6], e.g., all one-loop corrections of the induced charge J_0 are proportional to $e^2 B$. This is consistent with our work. Our result is at two-loop order; J_0 is proportional to $((e^4)/m)B$, where m is the Chern–Simons gauge mass. At the one-loop level, $J_0 \propto e^2$, there is no gauge propagator; at the two-loop level, $J_0 \propto e^4/m$, there is one. In the one-loop correction, if there are monopoles, there is the Dirac quantization condition of magnetic charge g, $eg/2\pi = n, n = 1, 2, \cdots$. In the two-loop correction, if there are monopoles, there is the Dirac quantization condition of magnetic charge, and also the quantization of Chern–Simons, $eg/2\pi = n$ and $4\pi m/e^2 = n$. Thus it can be seen that we have established a relation between the Dirac quantization condition of magnetic charge and the quantization of the Chern–Simons coefficient in QED of 2+1 space-time dimensions, with the Chern–Simons topological mass term in the presence of any magnetic monopole with magnetic charge g. It is easy to see from (31) that the Dirac quantization condition of magnetic charge leads to the quantization of the Chern–Simons coefficient, or conversely, the requirement of the quantization of the Chern–Simons coefficient so that the nontrivial gauge invariance can be kept leads to the Dirac quantization condition of magnetic charge. This demonstrates that the Chern–Simons coefficient must be also quantized in the Abelian case in the existence of a topological charge.

4 Conclusions

By perturbative calculations of high-temperature ground-state current, an additional anomalous three-dimensional Chern–Simons topological mass term is induced in the conventional effective action. This approach is different from the four-dimensional theory in that the anomaly in three dimensions appears just in the ground-state current, rather than in the divergence of the ground state current. This implies that not only must the anomalous term violate parity symmetry, but it also produces a mass for the three-dimensional gauge fields. In the three-dimensional electromagnetic fields case, only the vacuum polarization graph contributes to the high-temperature ground-state current, and the induced anomalous term ($\sim \epsilon^{ijl}F_{ij}A_l$) implies that the propagator of a fermion coupled to the massive vacuum polarization gauge bosons behaves like a boson [26]. In non-Abelian case, the contributions to the high-temperature ground-state current come from not only the vacuum polarization graph but also the two triangle graphs. We have also obtained the relation between the quantization of the Chern–Simons coefficient and the Dirac quantization condition of magnetic charge. This relation implies that the Dirac quantization condition of magnetic charge leads to the quantization of the Chern–Simons coefficient in a (2+1)-dimensional QED with the Chern–Simons topological mass term, so the Chern–Simons coefficient must be also quantized in the presence of a magnetic monopole, just as in the non-Abelian case.

The computations of the induced Chern–Simons term in three dimensions beginning from the high-temperature limit of a four-dimensional theory with an axial anomaly are different from those in Redlich's work [6]. Our approach is also different than other authors' previous works [18,20]. Thus there are some significant results.

References

1. S. Adler, Phys. Rev. **177**, 2426 (1969); J.S. Bell and R.Jackiw, Nuov. Cim. A **60**, 47 (1969)
2. W.A. Bardeen, Phys. Rev. **184**, 1848 (1969); Y. Cai and H.Y. Guo, Commun. Theor. Phys. **29**, 111 (1998)
3. J.I. Kapusta, *Finite-Temperature Field Theory* (Cambridge, London 1989); N.P. Landsman and Ch.G. van Weert, Phys. Rep. **145**, 141 (1987)
4. W. Siegel, Nucl. Phys. B **156**, 135 (1979); J. Schonfeld, Nucl. Phys. B **185**, 157 (1981)
5. S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); *ibid.*, Ann. Phys. (N.Y.) **140**, 372 (1982)
6. A.N. Redlich, Phys. Rev. Lett. **52**, 18 (1984); *ibid.*, Phys. Rev. D **29**, 2366 (1984)
7. R. Jackiw and S. Templeton, Phys. Rev. D **23**, 2291 (1981)
8. R. Pisarski, Phys. Rev. D **34**, 3851 (1986)
9. J. Schwinger, Phys. Rev. **82**, 664 (1951)
10. T. Appelquist and J. Carazzone, Phys. Rev. D **11**, 2856 (1975)
11. D. Gross, R. Pisarski, and L. Yaffe, Rev. Mod. Phys. **53**, 43 (1981)
12. G. ‘t Hooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972); P. Breitenlohner and D. Maison, Comm. Math. Phys. **52**, 11 (1977)
13. W. Bardeen, R. Gastmans, and B. Lautrup, Nucl. Phys. B **46**, 319 (1972)
14. Z.H. Wang and H.Y. Guo, Commun. Theor. Phys. **21**, 361 (1994)
15. M.B. Kislinger and P.D. Morley, Phys. Rev. D **13**, 2771 (1976); W.Y. Zhao, Nuov. Cim. A **76**, 525 (1983)
16. M. Atiyah and I. Singer, Ann. Math. **87**, 484 (1968)
17. G. Dunne, K. Lee, and C. Lu, Phys. Rev. Lett. **78**, 3434 (1997); S. Deser, L. Griguolo, and D. Seminara, Phys. Rev. Lett. **79**, 1976 (1997); C.D. Fosco, G.L. Rossini, and F.A. Schaposnik, Phys. Rev. Lett. **79**, 1980 (1997)
18. N. Bralic, C.D. Fosco, and F.A. Schaposnik, Phys. Lett. B **383**, 199 (1996)
19. R. Jackiw and S.Y. Pi, Phys. Lett. B **423**, 364 (1998); Phys. Rev. D **56**, 6547 (1997)
20. F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983); M. Henneaux and C. Teitelboim, Phys. Rev. Lett. **56**, 689 (1986); J. Dunne, R. Jackiw, and C. Trugenberger, Ann. Phys. (N.Y.) **194**, 197 (1989)
21. P.A.M. Dirac, Proc. Roy. Soc. A **133**, 60 (1931)
22. W. Chen, G.W. Semenoff, and Y.S. Wu, Phys. Rev. D **46**, 5521 (1992)
23. R.D. Pisarski and S. Rao, Phys. Rev. D **32**, 2081 (1985)
24. A. Polyakov, Mod. Phys. Lett. A **3**, 325 (1988)
25. S. Coleman and B. Hill, Phys. Lett. B **159**, 184 (1985)
26. G.W. Semenoff, Phys. Rev. Lett. **62**, 715 (1988); W. Chen, Phys. Lett. B **251**, 415 (1990); V.P. Spiridonov and F.V. Tkachov, B **260**, 109 (1991)